

§7 HERMITIAN LINE BUNDLES

Notiztitel

Version 1.0

As before, M is a smooth manifold and $\pi: L \rightarrow M$ denotes a complex line bundle over M .

(7.1) DEFINITION: A HERMITIAN LINE BUNDLE is a line bundle for which every fibre $L_a, a \in M$, has a Hermitian metric depending smoothly on the points $l \in L$.

The Hermitian metric can be given by a map

$$H: L \times L \rightarrow \mathbb{C}$$

such that $H|_{L_a \times L_a}: L_a \times L_a \rightarrow \mathbb{C}$ is a Hermitian scalar product for all $a \in M$. And the smoothness condition simply means that H is smooth. We denote $H(l, l') = \langle l, l' \rangle, l, l' \in L$.

In case of the trivial line bundle $L = M \times \mathbb{C}$ we obtain a HERMITIAN METRIC H_0 by setting

$$H_0((a, z), (a, w)) = \langle (a, z), (a, w) \rangle := \bar{z}w, z, w \in \mathbb{C}, a \in M.$$

H_0 is called the CONSTANT Hermitian metric. We see that any other Hermitian metric on L is given by a smooth $h: M \rightarrow \mathbb{R}, h(a) > 0$, for all $a \in M$ by

$$H(l, l') := h(a) \langle l, l' \rangle = h(a) H_0(l, l')$$

i.e.

$$H((a, z), (a, w)) = h(a) \bar{z}w.$$

(7.2) LEMMA: A hermitian line bundle (L, H) such that the underlying line bundle is the trivial line bundle $M \times \mathbb{C}$ is isomorphic to the trivial line bundle with the constant Hermitian metric H_0 .

□ Proof. The general case is of the form

$$H((a, z), (a, w)) = h(a) \bar{z} w,$$

and $\Phi: M \times \mathbb{C} \rightarrow M \times \mathbb{C}$, $(a, z) \mapsto (a, \sqrt{h(a)} z)$,

defines an isomorphism of hermitian line bundles $(M \times L, H)$, $(M \times L, H_0)$: Φ is an isomorphism of line bundles with $H(l, l') = H_0(\Phi(l), \Phi(l'))$, $l, l' \in M \times L$. □

From the local existence we can conclude that on every line bundle over a paracompact manifold there exists a Hermitian metric H such that it is a Hermitian line bundle.

To a Hermitian line bundle (L, H) one associates the circle bundle $L^1 \rightarrow M$, where

$$L^1 := \{ l \in L : H(l, l) = 1 \}.$$

This is a principal fibre bundle with the circle group $S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ as its structure group. Conversely, if $P \rightarrow M$ is a principal fibre bundle with structure group S^1 and $\rho: S^1 \rightarrow \mathbb{C}^\times = GL(1, \mathbb{C})$ is the natural representation $\rho(z) = z: \mathbb{C} \rightarrow \mathbb{C}$, $w \mapsto zw$, then the

associated vector bundle $L = P \times_{\mathbb{S}^1} \mathbb{C}$ is a line bundle, where \mathbb{S}^1 acts by scalar multiplication. The Hermitian metric H on L is then given by

$$H([x, z], [y, w]) := \bar{z}w, \text{ where } x, y \in P, z, w \in \mathbb{C}.$$

(7.3) PROPOSITION: The group of isomorphism classes of Hermitian line bundles (L, H) over M is isomorphic to $H^1(M, \mathbb{S}^1) \cong H^2(M, \mathbb{Z})$.

Here, $H^1(M, \mathbb{S}^1)$ is Čech cohomology with respect to the sheaf of \mathbb{S}^1 -valued smooth germs of functions on M (see end of section 3).

We now study line bundles on which there exists a connection together with a Hermitian metric.

(7.4) DEFINITION: Given a Hermitian line bundle (L, H) over M , a connection ∇ on L is called COMPATIBLE with H if for all sections $s, t \in \Gamma(U, L)$ and all vector fields $X \in \mathfrak{X}(U)$, $U \subset M$ open, we have

$$L_X \langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle$$

$$(\langle s, t \rangle = H(s, t)).$$

Such a connection is also called a Hermitian connection.

(7.5) PROPOSITION: A connection ∇ on L is compatible with a Hermitian metric H , if and only if the local gauge potentials $(\alpha_j)_{j \in I}$ with respect to local trivializations $\varphi_j: L|_{U_j} \rightarrow U_j \times \mathbb{C}$, $\cup U_j = M$, can be chosen to be real one forms $\alpha_j \in \Omega^1(M, \mathbb{R})$.

□ Proof. A collection of trivializations $\varphi_j: L|_{U_j} \rightarrow U_j \times \mathbb{C}$ can be chosen in such a way that φ_j is an isomorphism of hermitian line bundles with respect to the constant Hermitian metric H_0 on $U_j \times \mathbb{C}$, cf. Lemma (7.2). Now, $s, t \in \Gamma(U_j, L)$ have the form

$$s = fs_j, \quad t = gs_j, \quad g, f \in \mathcal{E}(U_j),$$

where $s_j(a) = \varphi_j^{-1}(a, 1)$. Hence,

$$\langle s, t \rangle = \langle (a, f(a)), (a, g(a)) \rangle_0 = \bar{f}(a)g(a), \quad \text{i.e.}$$

$$L_X \langle s, t \rangle = (L_X \bar{f})g + \bar{f} L_X g.$$

$$\begin{aligned} \langle \nabla_X s, t \rangle &= \langle (a, L_X f(a) + 2\pi i \alpha_j(X) f(a)), (a, g(a)) \rangle \\ &= (\overline{L_X f} - 2\pi i \bar{\alpha}_j(\bar{X}) \bar{f}(a)) g(a) \end{aligned}$$

$$\langle s, \nabla_X t \rangle = \bar{f}(a) (L_X g(a) + 2\pi i \alpha_j(X) g(a)).$$

Compatibility is therefore equivalent to

$$(L_X \bar{f})g + \bar{f} L_X g = \overline{L_X f} + \bar{f} L_X g - 2\pi i (\bar{\alpha}_j(\bar{X}) - \alpha_j(X)) \bar{f}g.$$

If we restrict to real vector fields this equation for

all f, g amounts to

$$0 = \bar{\alpha}_j(x) - \alpha_j(x).$$

Hence α_j is a real one form.

The converse can be read off the above formulas. \square

Another characterization is the following:

(7.6) PROPOSITION:^[*] A connection ∇ on L is compatible with a Hermitian metric H , if and only if for all non zero sections $s \in \Gamma(U, L^x)$, $U \subset M$ open:

$$d(H(s, s)) = 2H(s, s) \operatorname{Re} \left(\frac{\nabla s}{s} \right),$$

where $\frac{\nabla s}{s}$ denotes the one form $\beta \in \Omega^1(U, \mathbb{C})$ given by

$$\nabla_X s = \beta(X) s, \quad X \in \mathcal{D}(U).$$

The compatibility is therefore also equivalent to

" If $s \in \Gamma(U, L^x)$ is of length 1, i.e. $H(s, s) \equiv 1$, then $\frac{\nabla s}{s}$ is purely imaginary " ^[*]

Concerning the existence of compatible connections we conclude:

* Wrong: Proof!

(7.7) COROLLARY: Let (L, H) be a Hermitian line bundle. Then there exists a compatible connection ∇ . Given such a compatible connection ∇ the set of all connections compatible with H is the affine space

$$\nabla + 2\pi i \Omega^1(M, \mathbb{R}).$$

□ Proof. In the existence discussion in section 4 one only has to make sure to choose the one forms on the trivializations $L_{U_j} \rightarrow U_j \times \mathbb{C}$ as real one forms. The second statement follows from the description of all connections on L as $\nabla + \Omega^1(M, \mathbb{C})$. □

A given ∇ on the other hand may not be compatible to any metric H .

The curvature $\Omega = \text{Curv}(\nabla, L)$ of a connection on L compatible with a given Hermitian metric is always a real two form $\Omega \in \Omega^2(M, \mathbb{R})$.